Tutorial 4: Selected problems of Assignment 3

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§ 6.4 Taylor's Theorem

Recall Taylor's Theorem (Thm 6.4.1)

Thm (Taylor's Theorem) Let $f:[a,b]\to\mathbb{R}$ be a function such that $f, f', \dots, f^{(n)}$ are continuous on $[a,b]$ and $f^{(n+1)}$ exists on (a,b). Then for each fixed $x_0 \in [a,b]$, for any $x \in [a,b]$, there exists c between x and x_0 such that

$$f(x) = T_n(x) + R_n(x)$$

where $T_n(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

is the n^{th} Taylor polynomial of f at x_0, and

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$ is the Lagrangian form of the remainder

(§6.4 Q9)

Q1) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x$

Show that for each fixed $x_0 \in \mathbb{R}$, for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} R_n(x) = 0$$

Sol. For each fixed $x_0 \in \mathbb{R}$, $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

we would like to compute $R_n(x)$

Note that $f(x) = \sin x$; $f'(x) = \cos x$

$f''(x) = -\sin x$; $f^{(3)}(x) = -\cos x$

$f^{(4)}(x) = \sin x$ ...

Therefore, for each $m \in \mathbb{N}$,

$$f^{(m)}(x) = \begin{cases} (-1)^k \sin x & \text{, if } m = 2k \text{ for some } k \in \mathbb{N} \\ (-1)^{k-1} \cos x & \text{, if } m = 2k-1 \text{ for some } k \in \mathbb{N} \end{cases}$$

Hence we have 2 cases:

Case 1: $n = 2k$, $\exists k \in \mathbb{N}$: then

$$R_n(x) = \frac{f^{(2k+1)}(c)}{(2k+1)!}(x-x_0)^{2k+1} = \frac{(-1)^k \cos c}{(2k+1)!}(x-x_0)^{2k+1}$$

for some c between x and x_0. Then

$$|R_n(x)| = \left|\frac{(-1)^k \cos c}{(2k+1)!}(x-x_0)^{2k+1}\right| \leq \frac{|x-x_0|^{2k+1}}{(2k+1)!} = \frac{|x-x_0|^{n+1}}{(n+1)!}$$

Case 2: $n = 2k-1$, $\exists k \in \mathbb{N} \cup \{0\}$ then

$$R_n(x) = \frac{f^{(2k)}(c)}{(2k)!}(x-x_0)^{2k} = \frac{(-1)^k \sin c}{(2k)!}(x-x_0)^{2k}$$

for some c between x and x_0. Then

$$|R_n(x)| = \left|\frac{(-1)^k \sin c}{(2k)!}(x-x_0)^{2k}\right| \leq \frac{|x-x_0|^{2k}}{(2k)!} = \frac{|x-x_0|^{n+1}}{(n+1)!}$$

Therefore, in any case, we have $|R_n(x)| \leq \dfrac{|x-x_0|^{n+1}}{(n+1)!}$

Note that $\lim\limits_{n} \dfrac{|x-x_0|^{n+1}}{(n+1)!} = 0$ (by Ratio test)

Therefore, by Sandwich Theorem, $\lim\limits_{n\to+\infty} R_n(x) = 0$

(§6.4 Q10)

Q2) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & , \; x \neq 0 \\ 0 & , \; x = 0 \end{cases}$$

Show that $f^{(n)}(0) = 0$, for all $n \in \mathbb{N}$

Hence show that $\lim_{n\to\infty} R_n(x) \neq 0$ at $x_0 = 0$, for all x

Sol Note that for $x \neq 0$, $f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}}$;

$$f''(x) = -\frac{6}{x^4} e^{-\frac{1}{x^2}} + \frac{2}{x^3} \cdot \frac{2}{x^3} e^{-\frac{1}{x^2}} = \left(-\frac{6}{x^4} + \frac{4}{x^6}\right) e^{-\frac{1}{x^2}}$$

$\therefore$ We would like to show the following claim :

Claim 1 $\forall n \in \mathbb{N}$, there exists a polynomial $P_n(y)$

with positive leading coefficients

i.e. $P_n(y) = a_{m,n} y^m + \cdots + a_{0,n}$, where $a_{m,n} > 0$

and $f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$, $\forall x \neq 0$

Pf of claim 1 Induction on n :

Base step : $n=1$; Chose $p_1(x) = 2x^3$, then

$$f'(x) = \frac{2}{x^3} e^{-\frac{1}{x^2}} = p_1\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} \quad \therefore n=1 \text{ is true}$$

Inductive step: Suppose the claim holds for $n=N$:

$$f^{(N)}(x) = P_N\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}} , \quad \forall x \neq 0$$

when $n=N+1$, differentiate the above equation yields

$$f^{(N+1)}(x) = \left(P_N\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}\right)'$$

$$= P_N'\left(\frac{1}{x}\right)\cdot\left(\frac{-1}{x^2}\right) e^{-\frac{1}{x^2}} + P_N\left(\frac{1}{x}\right)\left(\frac{2}{x^3} e^{-\frac{1}{x^2}}\right)$$

$$= \left(-\frac{1}{x^2}\cdot P_N'\left(\frac{1}{x}\right) + \frac{2}{x^3} P_N\left(\frac{1}{x}\right)\right) e^{-\frac{1}{x^2}}$$

Hence, set $P_{N+1}(x) = -\frac{1}{x^2} P_N'\left(\frac{1}{x}\right) + \frac{2}{x^3} P_N\left(\frac{1}{x}\right)$,

Note that $2y^3 P_N(y)$ has higher degree than $-y^2 P_N'(y)$

and $P_N(x)$ has positive leading coefficient, $\therefore$ $P_{N+1}(x)$ has

positive leading coefficient. $\therefore$ the claim is true for $n=N+1$

Therefore, $f^{(n)}(x) = P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x^2}}$, $\forall n \in \mathbb{N}$, $\forall x \neq 0$ — ▨

We then prove the main question:

Claim 2: $\forall n \in \mathbb{N}, \quad f^{(n)}(0) = 0$

Pf of claim 2:

Base step: $n=1$; $h'(0) = \lim_{x\to 0} \dfrac{h(x) - h(0)}{x - 0}$

$= \lim_{x\to 0} \dfrac{e^{-\frac{1}{x^2}}}{x}$; Let $y = \frac{1}{x}$, then $x\to 0 \Leftrightarrow y\to\infty$

$= \lim_{y\to\infty} \dfrac{y}{e^{y^2}} = \lim_{y\to\infty} \dfrac{1}{2ye^{y^2}}$ (L'Hospital's Rule) $= 0$

$\therefore$ The claim is true for $n=1$

Inductive step: Suppose the claim is true for $n=N$,

$$h^{(n)}(0) = 0$$

When $n = N+1$, $h^{(N+1)}(0) = \lim_{x\to 0} \dfrac{h^{(N)}(x) - h^{(N)}(0)}{x - 0}$

$= \lim_{x\to 0} \dfrac{P_N(\frac{1}{x}) e^{-\frac{1}{x^2}}}{x}$ (by claim 1)

Let $y = \frac{1}{x}$, then $x \to 0 \Leftrightarrow y \to \infty$

$$\therefore = \lim_{y\to\infty} \frac{y\, p_n(y)}{e^{y^2}} = 0$$

by noting that $e^{y^2} = 1 + y^2 + \cdots + \frac{y^{2n}}{n!} + R_n(y)$

$\geqslant y^2 \cdot p_n(y)$ for sufficiently large y

and hence $\underbrace{0 \leqslant}_{} \frac{y\, p_n(y)}{e^{y^2}} \leqslant \frac{1}{y}$ for sufficiently large y

($\because$ p_n has positive leading coefficient)

$\therefore$ The claim is true for $n = N+1$.

Hence, $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$

Finally, apply Taylor's Theorem at $x_0 = 0$ for each $n \in \mathbb{N}$

$f(x) = T_n(x) + R_n(x) = R_n(x)$ as $f^{(k)}(0) = 0$, $\forall k \in \mathbb{N}$

$$\therefore \lim_{n\to\infty} R_n(x) = f(x) = e^{-\frac{1}{x^2}} \neq 0 \text{ for all } x \in \mathbb{R}.$$

(Supplementary Exercises Q2)

Q3) Show that for $x \in [-\frac{1}{2}, 1]$,

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Sol Let $f : (-1, +\infty) \to \mathbb{R}$ be defined by

$$f(x) = \log(1+x)$$

Note that $f'(x) = \frac{1}{1+x}$; $f''(x) = -\frac{1}{(1+x)^2}$, $f'''(x) = \frac{2}{(1+x)^3}$

In general, we see that $f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k}$, $\forall k \in \mathbb{N}$

$\therefore$ At $x_0 = 0$, $T_n(x) = f(0) + \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k$

and $R_n(x) = \frac{(-1)^n x^{n+1}}{(n+1)(1+c)^{n+1}}$. c between 0 and x

By Taylor's theorem, for all $x \in (-1, +\infty)$, $n \in \mathbb{N}$,

$$f(x) = \log(1+x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k + \frac{(-1)^n x^{n+1}}{(n+1)(1+c)^{n+1}}$$

We then study when does $R_n(x) \to 0$ as $n \to \infty$;

Claim $\lim_{n\to\infty} R_n(x) = 0$ for $x \in [-\frac{1}{2}, 1]$

Pf of Claim $R_n(x) = \frac{(-1)^n x^{n+1}}{(1+c)^{n+1}(n+1)}$

Case 1: $x \in [0,1]$; then $|x| \leq 1$;

Also, $c \in [0,x] \Rightarrow c \geq 0 \Rightarrow \frac{1}{1+c} \leq 1$

$\therefore \left| R_n(x) \right| \leq \frac{1}{n+1} \Rightarrow \lim_{n\to\infty} |R_n(x)| = 0$

and hence $\lim_{n\to\infty} R_n(x) = 0$ Corrected version

Case 2: $x \in [-\frac{1}{2}, 0]$; then $|x| \leq \frac{1}{2}$

Also, $c \in [x, 0] \Rightarrow -\frac{1}{2} \leq c \Rightarrow 1+c \geq \frac{1}{2}$;

$\therefore \left(\frac{|x|}{1+c}\right)^{n+1} \leq \left(\frac{\frac{1}{2}}{\frac{1}{2}}\right)^{n+1} = 1$

$\therefore |R_n(x)| \leq \frac{1}{(n+1)}$ and similarly $\lim_{n} R_n(x) = 0$ – ▨

Therefore, for $x \in [-\frac{1}{2}, 1]$, $\log(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^{k-1} x^k}{k}$

(Supplementary Exercises 3)

Q4) Express $-12+x^2+3x^4$ in the form

$$a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4$$

Sol) Let $f(x) = -12 + x^2 + 3x^4$; then

$f'(x) = 2x + 12x^3$; $f''(x) = 2 + 36x^2$;

$f'''(x) = 72x$; $f^{(4)}(x) = 72$

At $x_0 = 1$,

$$T_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$$

$$= -8 + 14(x-1) + 19(x-1)^2 + 12(x-1)^3 + 3(x-1)^4$$

$$R_4(x) = \frac{f^{(5)}(c)}{5!}(x-1)^5 = 0$$

$\therefore$ By Taylor's Theorem,

$$f(x) = T_4(x) + R_4(x) = -8 + 14(x-1) + 19(x-1)^2 + 12(x-1)^3 + 3(x-1)^4$$

(Supp. Ex. 4)

Q5) Let $f: \mathbb{R} \to \mathbb{R}$ be infinitely differentiable function

If a polynomial $p(x) = a_n x^n + \cdots + a_0$, $\exists\, C > 0$ $\exists\, x_0 \in \mathbb{R}$ s.t.

$$|f(x) - p(x)| \leq C|x - x_0|^{n+1}, \text{ for all } x \in V_\delta(x_0)$$

then $p(x) = T_n(x)$, where $T_n(x)$ is n^{th}-Taylor polynomial at x_0.

Sol By Taylor's Theorem, $f(x) = T_n(x) + R_n(x)$

$\therefore$ for all $x \in V_\delta(x_0)$,

$$|T_n(x) - p(x)| \leq |T_n(x) - f(x)| + |f(x) - p(x)|$$

$$\leq |R_n(x)| + C|x - x_0|^{n+1}$$

$$= \left(\frac{|f^{(n+1)}(c)|}{(n+1)!} + C \right) |x - x_0|^{n+1} \leq (C' + C)\, |x - x_0|^{n+1}$$

where $C' = \dfrac{\sup_{y \in \overline{V_\delta}(x_0)} |f^{(n+1)}(y)|}{(n+1)!}$. Write $C'' = C' + C$

$\therefore \quad |T_n(x) - p(x)| \leq C'' |x - x_0|^{n+1}, \quad \forall x \in V_\delta(x_0).$

On the other hand. Write $T_n(x) = b_n(x-x_0)^n + \cdots + b_0$

then $|T_n(x) - p_n(x)| = |(b_n - a_n)(x-x_0)^n + \cdots + (b_0 - a_0)|$

$\therefore \quad |(b_n - a_n)(x-x_0)^n + \cdots + (b_0 - a_0)| \leq C'' |x - x_0|^{n+1}$

substitute $x = x_0$ above, $|b_0 - a_0| \leq 0$, i.e. $a_0 = b_0$

$\therefore \quad |(b_n - a_n)(x-x_0)^n + \cdots + (b_1 - a_1)(x - x_0)| \leq C'' |x - x_0|^{n+1}$

divide both sides by $(x - x_0)$ yields

$$|(b_n - a_n)(x-x_0)^{n-1} + \cdots + (b_1 - a_1)| \leq C'' |x - x_0|^n$$

Again, substitute $x = x_0$ gives $b_1 = a_1$

Iterate the above procedure until $|b_n - a_n| \leq C'' |x - x_0|$

and hence $b_j = a_j, \quad \forall j = 0, 1, \ldots, n$

$\therefore \quad p(x) = T_n(x)$